BASE VECTORS FOR SOLVING OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

The distributed parameters systems can be described by linear two-dimensional (dependent on two spatial directions) parabolic partial differential equations. Using the finite difference method a distributed parameters system can be transformed to a linear discrete state space model. The controller design based on this description is complicated because of the large dimension of the model. Therefore, a model reduction method has to be used. We transform the state space model to the balanced realization of the system and show that the state vector of the model can be expressed as the series of columns of the transformation matrix. These columns can be imaged as base vectors of the state space.

1 Introduction

There are many industrial processes that have distributed parameters behaviour. Consequently, these processes cannot be modelled by lumped inputs and/or lumped outputs models for correct representation.

This paper deals with two-dimensional dynamic processes (systems with parameters dependent on two spatial directions) that can be described by lumped inputs and distributed output models. These models can be mathematically described by partial differential equations (PDE) [5]. Unlike ordinary differential equations, the PDEs contain, in addition, derivatives with respect to spatial directions. Consequently, the partial differential equations lead to more accurate models but their complexity is larger.

The dynamic behaviour of the distributed parameters system, which is described by the PDE, can be approximately described by a finite-dimensional model, for example, by using the finite difference method [1]. Then the ordinary differential equation model with large dimension is obtained and can be used for a finite-dimensional controller design. Unfortunately, for online solving of an optimization problem, e.g. the model predictive control approach [6, 2], the large model dimension introduces a problem for the control design. Therefore, a model reduction method has to be used.

Partial differential equations are usually solved by base functions. This approach is based on that the solution of the PDEs can be expressed as series of the base functions. This paper presents similar methodology. The discrete state space description of a distributed parameters system is transformed to the balanced realization [8]. Then the system state can be expressed as the product of the transformation matrix and the balanced state. We can imagine the columns of the transformation matrix as base vectors to which the system state can be expanded. Then using only the first few base vectors corresponds to the truncation of states with smaller influence on the input/output behaviour of the system.

The paper is organized as follows. In section 2, the distributed parameters model for the finite controller design is developed. In section 3, the balanced truncation method is described. In section 4, the relationship between the balanced realization and the system state expansion to an orthonormal base vectors is developed. In section 5, this approach is demonstrated on a heat transfer process.

2 Distributed Parameter Process Description

In this section, the model of a heat transfer process described by a linear two-dimensional parabolic PDE [5] is developed for a finite-dimensional controller design. At first, the stationary PDE is transformed to a linear equation system using the finite difference approximation [1]. Then the implicit scheme [1] and this equation system are used for the transformation of the evolutionary PDE to a linear dynamic discrete system.

2.1 Stationary Partial Differential Equation

For the surface thermal conductivity λ [W/K] independent on the temperature Θ [K] and a surface heat source f [W/m^2], the heat transfer process in the stationary case can be described by a parabolic PDE

$$-\lambda\left(\frac{\partial^2\Theta(x,y)}{\partial x^2}+\frac{\partial^2\Theta(x,y)}{\partial y^2}\right)=-\lambda\,\Delta\Theta(x,y)=f(x,y), \tag{1}$$

where Δ is the Laplace operator. The unknown temperature Θ must satisfy equation (1) on an open set $\Omega=(0,L_1)\times(0,L_2)$ and boundary condition on $\partial\Omega$ ($\partial\Omega$ means the boundary of set Ω).

In this paper, the boundary condition which specifies the temperature gradient on the boundary $\partial\Omega$ is described by the following statement

$$-\lambda\frac{\partial\Theta(x,y)}{\partial n}=\alpha\big(\Theta(x,y)-\Theta_s(x,y)\big), \tag{2}$$

where n is the unit normal vector, α [W/(mK)] is an external heat transfer coefficient and Θ_s is the surrounding temperature. Note that equation (2) is known as Newton (or the third kind) boundary condition [5].

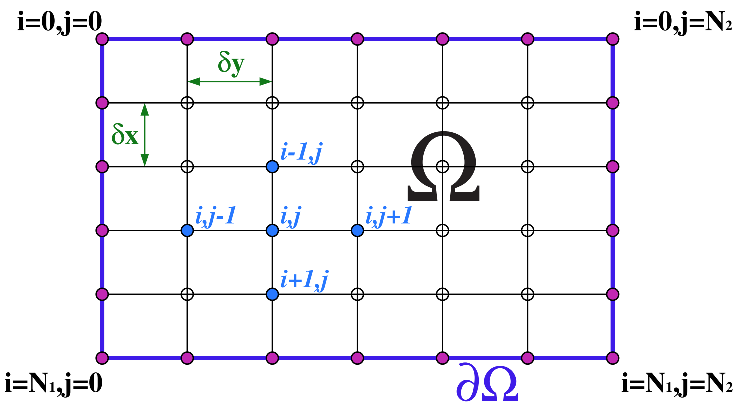

Figure 1: The mesh on the set Ω

For the transformation of PDE (1) with boundary condition (2) to the finite dimensional model, the set Ω is covered by an imaginary mesh so that the values of the mesh points satisfy $\boldsymbol{\Theta}_{i,j}=\Theta(i\,\delta x,j\,\delta y)$ on the closed set Ω, $\boldsymbol{F}_{i,j}=f(i\,\delta x,j\,\delta y)$ on the open set Ω and $\boldsymbol{F}_{i,j}=\boldsymbol{\Theta}_s(i\,\delta x,j\,\delta y)$ on the set $\partial\Omega$, where δx and δy are the grid sizes of the imaginary mesh and i, j are row and column indices, respectively (see Figure 1). Matrix $\boldsymbol{\Theta}$ is the matrix of the temperature values in the mesh points and matrix $\boldsymbol{F}$ represents the heat source f and the surrounding temperature Θ_s.

When we define vectors

$$\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\Theta}(\,:\,,0) \\ \boldsymbol{\Theta}(\,:\,,1) \\ \vdots \\ \boldsymbol{\Theta}(\,:\,,N_2-1) \\ \boldsymbol{\Theta}(\,:\,,N_2) \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} \boldsymbol{F}(\,:\,,0) \\ \boldsymbol{F}(\,:\,,1) \\ \vdots \\ \boldsymbol{F}(\,:\,,N_2-1) \\ \boldsymbol{F}(\,:\,,N_2) \end{bmatrix}, \tag{3}$$

$$\text{with indices} \quad l = j(N_1+1)+i,$$

where $\boldsymbol{\Theta}(:,0)$ means the zero column of the matrix $\boldsymbol{\Theta}$, $\boldsymbol{\Theta}(:,1)$ the first column and so on, partial differential equation (1) with Newton boundary condition equations (2) can be written in compact form

$$\boldsymbol{P}\boldsymbol{\theta} = \boldsymbol{f}. \tag{4}$$

More details were described in [7].

2.2 Evolutionary Partial Differential Equation

In the non stationary case, PDE (1) can be written as

$$\rho\, c_0 \frac{d\Theta(x,y,t)}{dt} - \lambda\, \Delta\Theta(x,y,t) = f(x,y,t), \tag{5}$$

where ρ [kg/m^2] is the surface density of the medium and c_0 [Ws/kgK] is its thermal capacity. In this case, the unknown temperature profile $\Theta(x,y,t)$, dependent on time t, must satisfy, for an initial condition $\Theta(x,y,t_0) = \Theta_{init}(x,y)$, equation (5) on the open set Ω and boundary condition (2) on $\partial\Omega$ for all time horizon $t \in \langle t_0,\, t_{end}\rangle$.

Using equation (4) and the implicit discretization scheme [1] with a sampling period δt, evolutionary PDE (5) with Newton boundary condition (2) can be approximated as

$$\begin{aligned} \boldsymbol{\theta}(k+1) &= \boldsymbol{M}\boldsymbol{\theta}(k) + \boldsymbol{N}\boldsymbol{f}(k), \qquad \boldsymbol{\theta}(k_0) = \boldsymbol{\theta}_{init}, \\ \boldsymbol{M} &= \left(\boldsymbol{I} + \frac{\delta t}{\rho\, c_0}\boldsymbol{P}\right)^{-1}, \quad \boldsymbol{N} = \left(\boldsymbol{I} + \frac{\delta t}{\rho\, c_0}\boldsymbol{P}\right)^{-1} \cdot \frac{\delta t}{\rho\, c_0}\,, \end{aligned} \tag{6}$$

where $\boldsymbol{I}$ is the identity matrix with the corresponding dimension.

3 Model Reduction Method

The accuracy of model (6) increases with decreasing grid sizes δx and δy. Unfortunately, for the advanced controller design, for example the predictive controller, a low dimension model is needed. In this section, one reduction method is shortly described.

3.1 Model Reduction by the Balanced Truncation Method

There are infinitely many different state space realizations for a given transfer function. But some realizations are more useful in control design. One of these realizations is the balanced realization which gives balanced Gramians for controllability $\boldsymbol{W}_c$ and observability $\boldsymbol{W}_o$ [8]. In addition, these Gramians are equal to the diagonal matrix $\boldsymbol{\Sigma}$

$$\boldsymbol{W}_c = \boldsymbol{W}_o = \boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1,\, \sigma_2,\, \dots,\, \sigma_n).$$

Note that the decreasingly ordered numbers,

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0,$$

are called the *Hankel singular values* of the system.

We suppose $\sigma_r \gg \sigma_{r+1}$ for some $r \in \langle 1; n)$. Then the balanced realization implies that the states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$ are less controllable and observable than the states corresponding to $\sigma_1, \ldots, \sigma_r$. The states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$ have smaller influence on the input/output behaviour of the system. Therefore, truncating those "less controllable and observable" states will not lose much information about the system input/output behaviour and the dimension of the model can be significantly reduced.

3.2 Transformation to the Balanced Realization

Model (6) can be expressed as

$$\begin{aligned}\boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{E}\boldsymbol{z}(k), \;\; \boldsymbol{x}(k_0) = \boldsymbol{x}_0, \\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k), \end{aligned} \tag{7}$$

where $\boldsymbol{x} = \boldsymbol{\theta}$ is a state vector of the model (temperature profile), $\boldsymbol{y}$ is its output vector (temperature measured in several points on the set Ω), $\boldsymbol{u}$ is its input vector (manipulated variable), $\boldsymbol{z}$ represents the surrounding temperature profile (measurable disturbance) and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$ are state matrices.

Gramians for controllability $\boldsymbol{W}_c$ and observability $\boldsymbol{W}_o$ [4] of model (7) can be expressed by

$$\boldsymbol{W}_c = \sum_{i=0}^{\infty} \boldsymbol{A}^i \left[\boldsymbol{B}, \boldsymbol{E}\right] \left[\boldsymbol{B}, \boldsymbol{E}\right]^T \left(\boldsymbol{A}^T\right)^i, \tag{8}$$

$$\boldsymbol{W}_o = \sum_{i=0}^{\infty} \left(\boldsymbol{A}^T\right)^i \boldsymbol{C}^T \boldsymbol{C}\, \boldsymbol{A}^i. \tag{9}$$

Model (7) can be transformed, by a matrix $\boldsymbol{Q}$

$$\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{v}, \tag{10}$$

to another realization

$$\begin{aligned}\boldsymbol{v}(k+1) &= \overline{\boldsymbol{A}}\boldsymbol{v}(k) + \overline{\boldsymbol{B}}\boldsymbol{u}(k) + \overline{\boldsymbol{E}}\boldsymbol{z}(k), \;\; \boldsymbol{v}(k_0) = \boldsymbol{v}_0, \\ \boldsymbol{y}(k) &= \overline{\boldsymbol{C}}\boldsymbol{v}(k) + \overline{\boldsymbol{D}}\boldsymbol{u}(k), \end{aligned} \tag{11}$$

where the matrices satisfy $\overline{\boldsymbol{A}} = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$, $\overline{\boldsymbol{B}} = \boldsymbol{Q}^{-1}\boldsymbol{B}$, $\overline{\boldsymbol{E}} = \boldsymbol{Q}^{-1}\boldsymbol{E}$, $\overline{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{Q}$, $\overline{\boldsymbol{D}} = \boldsymbol{D}$. Note that Gramians of the transformed realization (11) are equal to

$$\overline{\boldsymbol{W}_c} = \boldsymbol{Q}^{-1}\boldsymbol{W}_c(\boldsymbol{Q}^{-1})^T, \qquad \overline{\boldsymbol{W}_o} = \boldsymbol{Q}^T\boldsymbol{W}_o\boldsymbol{Q}\,, \tag{12}$$

$$\Longrightarrow \qquad \overline{\boldsymbol{W}_c}\,\overline{\boldsymbol{W}_o} = \boldsymbol{Q}^{-1}\boldsymbol{W}_c\boldsymbol{W}_o\boldsymbol{Q}\,. \tag{13}$$

Now we have to find transformation matrix $\boldsymbol{Q}$, such that realization (11) is the balanced realization of the system. Since Gramians are symmetric positive definite matrices, we can factor Gramian for observability as $\boldsymbol{W}_o = \boldsymbol{P}^T\boldsymbol{P}$, where matrix $\boldsymbol{P}$ is the Cholesky factor of matrix $\boldsymbol{W}_o$. Then equation (13) can be written as $\boldsymbol{Q}^{-1}\boldsymbol{W}_c\boldsymbol{P}^T\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{\Sigma}^2$. Using elementary rearrangement, the equation can be obtained as

$$\boldsymbol{Q}^{-1}\boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{W}_c\boldsymbol{P}^T\boldsymbol{P}\boldsymbol{Q} = \left(\boldsymbol{P}\boldsymbol{Q}\right)^{-1}\boldsymbol{P}\boldsymbol{W}_c\boldsymbol{P}^T\left(\boldsymbol{P}\boldsymbol{Q}\right) = \boldsymbol{\Sigma}^2. \tag{14}$$

Equation (14) means that $\boldsymbol{P}\boldsymbol{W}_c\boldsymbol{P}^T$ is similar to $\boldsymbol{\Sigma}^2$ and is positive definite. Therefore, there exists an orthonormal transformation matrix $\boldsymbol{U}$ ($\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$), such that

$$\boldsymbol{P}\boldsymbol{W}_c\boldsymbol{P}^T = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^T. \tag{15}$$

From equations (12)a and (15), we can derive the formula for the transformation matrix

$$\boldsymbol{Q} = \boldsymbol{P}^{-1}\boldsymbol{U}\boldsymbol{\Sigma}^{-1/2}. \tag{16}$$

Then model (11) is the balanced realization of model (7) and its Gramians for controllability and observability are $\overline{\boldsymbol{W}_c} = \overline{\boldsymbol{W}_o} = \boldsymbol{\Sigma}$ [8].

So if model (7) is a minimal realization and matrix $\boldsymbol{A}$ is stable, the transformation matrix $\boldsymbol{Q}$ can be obtained through the following procedure:

- compute the controllability and observability Gramians of model (7) $\boldsymbol{W}_c > 0$, $\boldsymbol{W}_o > 0$,
- find matrix $\boldsymbol{P}$ such that $\boldsymbol{W}_o = \boldsymbol{P}^T\boldsymbol{P}$,
- diagonalize $\boldsymbol{P}\boldsymbol{W}_c\boldsymbol{P}^T = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^T$ by using the singular value decomposition [3],
- let the $\boldsymbol{Q} = \boldsymbol{P}^{-1}\boldsymbol{U}\boldsymbol{\Sigma}^{-1/2}$.

4 Base Vectors

Considering the Hankel singular values are decreasingly ordered, the first state of state vector $\boldsymbol{v}$ has the biggest influence on the input/output behaviour of the system, the second state has less influence and so on. As it has been said in section 3, we can truncate states corresponding to small singular values σ_i and the model keeps enough information about the input/output behaviour of the system. In this section, we show that the system state vector can be expressed in an orthonormal base. Then using only the first r orthonormal base vectors (for expression of the system state vector) corresponds to the truncation of the states with smaller influence on the input/output behaviour of the system.

4.1 Decomposition of the System State Vector to the Balanced Base Vectors

Equation (10) can be rewritten as

$$\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{v} = \sum_{i=1}^{n} v_i\, \boldsymbol{q}_i\,, \tag{17}$$

where v_i are the elements of state vector $\boldsymbol{v}$ in balanced realization (11) and vectors $\boldsymbol{q}_i$ are the columns of matrix $\boldsymbol{Q} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n]$. Since the transformation matrix $\boldsymbol{Q}$ is not singular, the vectors $\boldsymbol{q}_i$ are linearly independent. So equation (17) means that state vector $\boldsymbol{x}$ can be expressed as the series of base vectors $\boldsymbol{q}_i$ that are multiplied by weighted coefficients v_i.

Note that if we want to truncate states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$, we express state vector $\boldsymbol{x}$ only by the first r base vectors $\boldsymbol{q}_i$

$$\boldsymbol{x} = \sum_{i=1}^{r} v_i\, \boldsymbol{q}_i\,.$$

4.2 Orthonormal Base Vectors

The base vectors $\boldsymbol{q}_i$ can be transformed to an orthonormal base by using the Gram-Schmidt orthonormalization. We decompose the matrix $\boldsymbol{Q}$ such that $\boldsymbol{Q} = \boldsymbol{O}\boldsymbol{R}$, where $\boldsymbol{R}$ is an upper triangular matrix and $\boldsymbol{O}$ is an orthonormal matrix ($\boldsymbol{O}^T\boldsymbol{O} = \boldsymbol{I}$). Then equation (17) can be rewritten as

$$\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{v} = \boldsymbol{O}\boldsymbol{R}\boldsymbol{v} = \boldsymbol{O}\tilde{\boldsymbol{v}} = \sum_{i=1}^{n} \tilde{v}_i\, \boldsymbol{o}_i\,, \tag{18}$$

where $\tilde{\boldsymbol{v}} = \boldsymbol{R}\boldsymbol{v}$ and vectors $\boldsymbol{o}_i$ are the columns of matrix $\boldsymbol{O} = \left[\boldsymbol{o}_1, \boldsymbol{o}_2, \ldots, \boldsymbol{o}_n\right]$. So equation (18) means that state vector $\boldsymbol{x}$ can be expressed by the orthonormal base vectors $\boldsymbol{o}_i$.

5 DEMONSTRATION EXAMPLE

Consider a heat transfer process in a furnace where $L_1 = L_2 = 0.9$ m described by equation (5) with constants $\lambda = 51$ W/K, $\rho = 2500$ kg/m^2, $c_0 = 1259$ Ws/(kg K) and $\alpha = 1.14$ W/(mK). The grid sizes are $\delta x = \delta y = 0.02$ m and the sampling period is $\delta t = 300$ s. The system has five lumped inputs (manipulated variables, see Figure 2) and the surface temperature is measured in 64 points which are uniformly distributed over the area Ω, see Figure 3b.

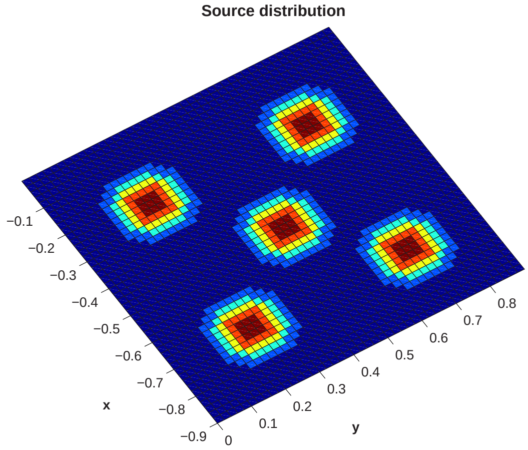

Figure 2: The heat source distribution $f(x, y)$

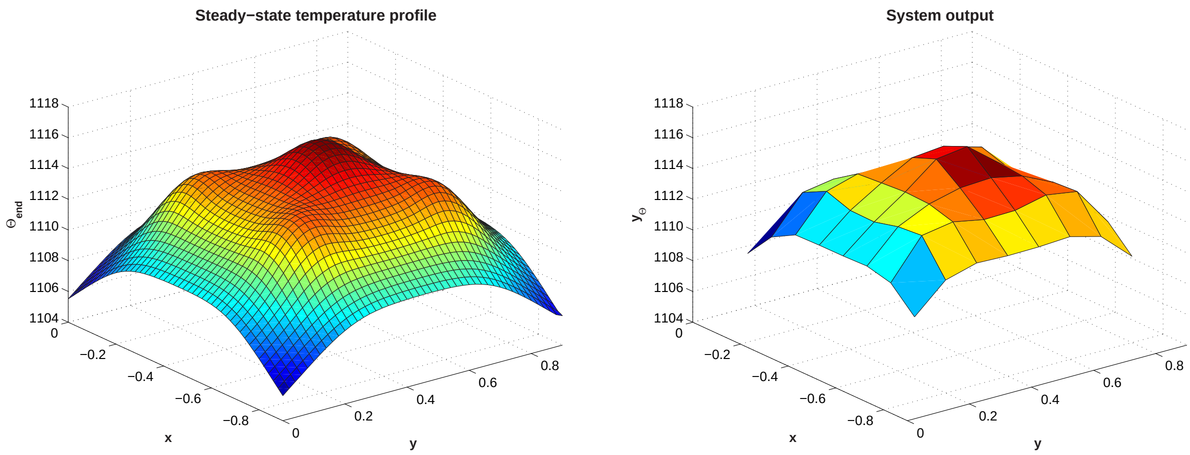

Figure 3: (a) The steady-state temperature distribution $\Theta(x, y)$; (b) The system output $\boldsymbol{y}$ (the temperature in 64 measurement points)

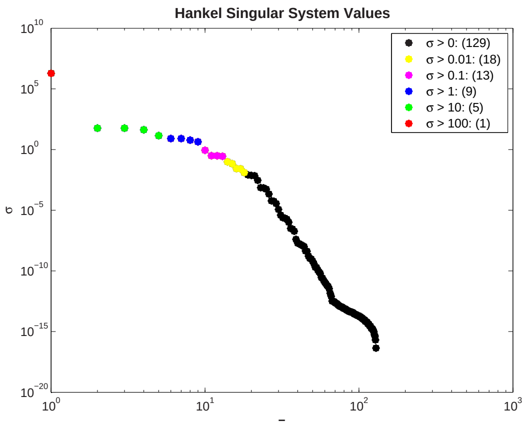

Figure 4: The Hankel singular values of the system

Figure 3a presents the steady-state temperature distribution (system state) for the unit step as input signal (see Figure 2) and the surrounding temperature $\Theta_s = 340$ K. Figure 3b shows the system output $\boldsymbol{y}$ – temperature in 64 measurement points. Figure 4 shows the Hankel singular values of the system.

Figure 5a presents the system output based on the first system state (in balanced realization), Figure 5b shows the base function corresponding to the first base vector (17) and Figure 5c shows the orthonormal base function corresponding to the first orthonormal base vector (18). Figure 6a presents the system output based on the first two system states and Figures 6b and 6c show the base function corresponding to the second base vector and the orthonormal base function corresponding to the second orthonormal base vector, respectively.

From Figures 5-14, it follows that the system state vector can be expressed in the series of the base functions (the columns of the transformation matrix) or the orthonormal base function, respectively. This methodology is similar to the analytical solution of PDEs based on the non-discrete base functions that are usually nonlinear even if the PDEs are linear.

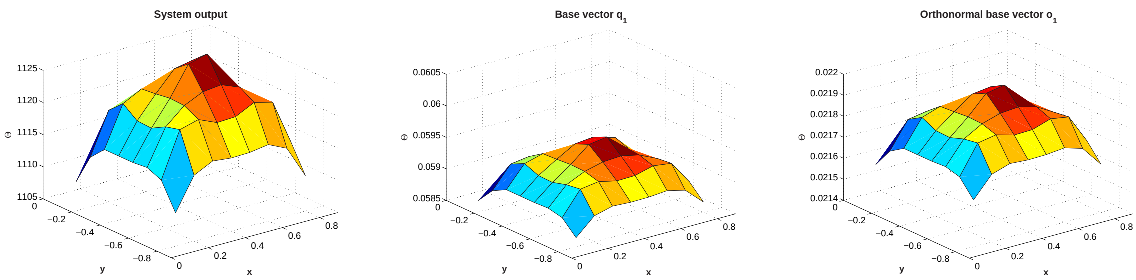

Figure 5: (a) Steady-state system output $\boldsymbol{y}$ based on the first state; (b) The first base function ($\boldsymbol{q}_1$); (c) The first orthonormal base function ($\boldsymbol{o}_1$)

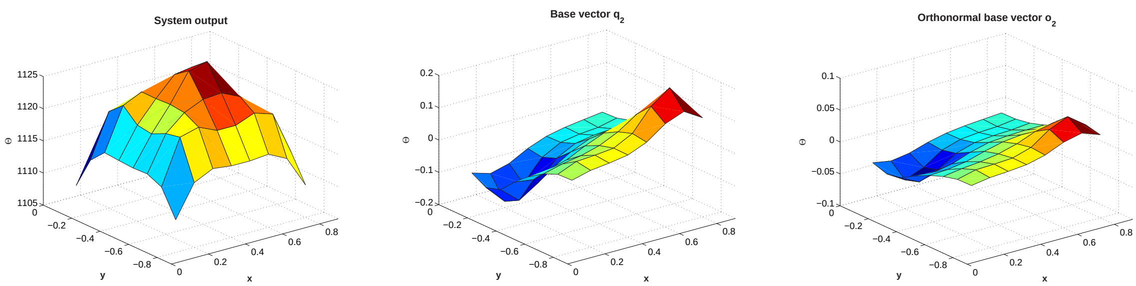

Figure 6: (a) Steady-state system output $\boldsymbol{y}$ based on the first two states; (b) The second base function ($\boldsymbol{q}_2$; (c) The second orthonormal base function ($\boldsymbol{o}_2$)

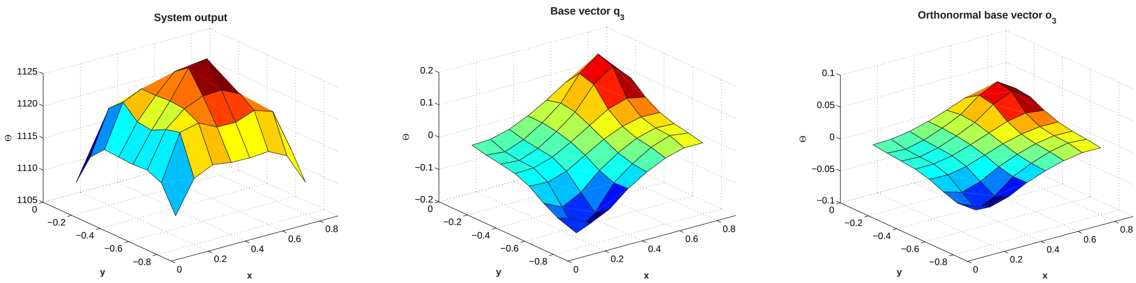

Figure 7: (a) Steady-state system output $\boldsymbol{y}$ based on the first three states; (b) The third base function ($\boldsymbol{q}_3$); (c) The third orthonormal base function ($\boldsymbol{o}_3$)

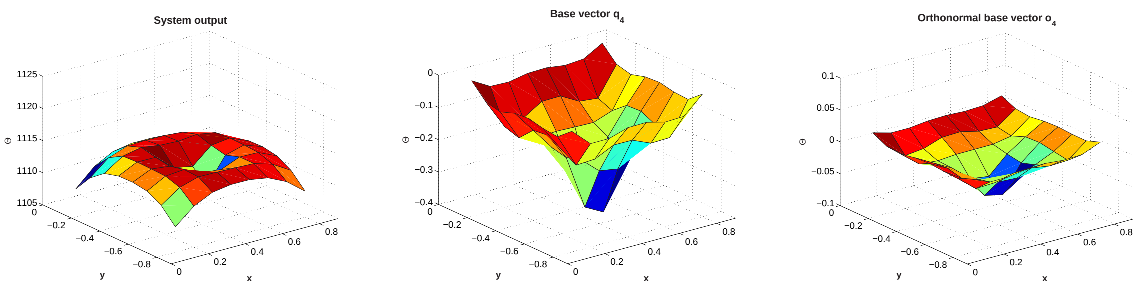

Figure 8: (a) Steady-state system output $\boldsymbol{y}$ based on the first four states; (b) The fourth base function ($\boldsymbol{q}_4$); (c) The fourth orthonormal base function ($\boldsymbol{o}_4$)

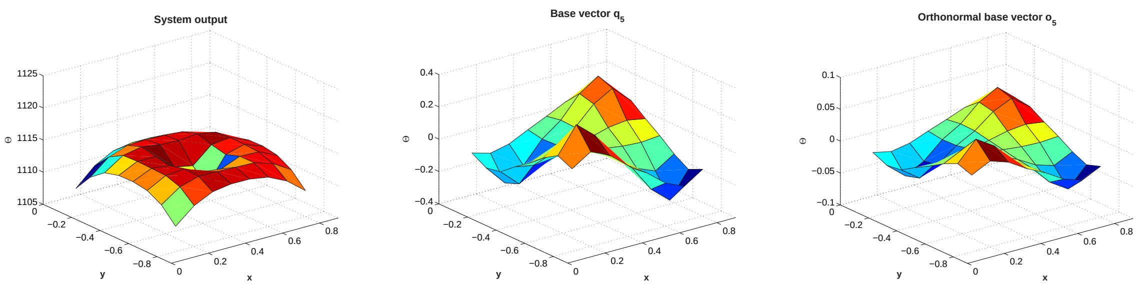

Figure 9: (a) Steady-state system output $\boldsymbol{y}$ based on the first five states; (b) The fifth base function ($\boldsymbol{q}_5$); (c) The fifth orthonormal base function ($\boldsymbol{o}_5$)

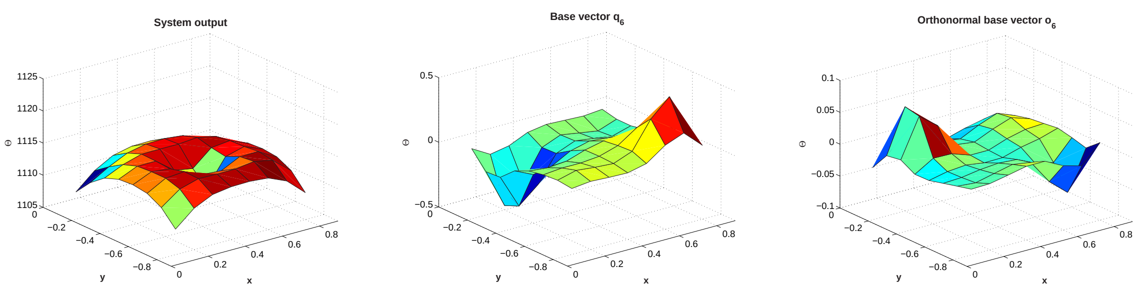

Figure 10: (a) Steady-state system output $\boldsymbol{y}$ based on the first six states; (b) The sixth base function ($\boldsymbol{q}_6$); (c) The sixth orthonormal base function ($\boldsymbol{o}_6$)

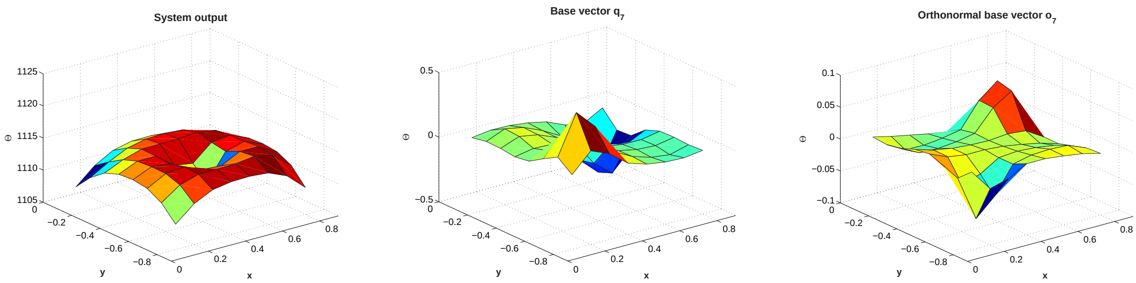

Figure 11: (a) Steady-state system output $\boldsymbol{y}$ based on the first seven states; (b) The seventh base function ($\boldsymbol{q}_7$); (c) The seventh orthonormal base function ($\boldsymbol{o}_7$)

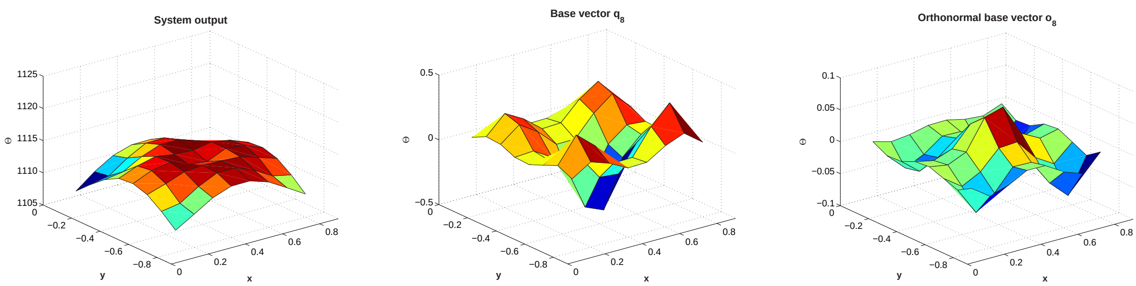

Figure 12: (a) Steady-state system output $\boldsymbol{y}$ based on the first eight states; (b) The eighth base function ($\boldsymbol{q}_8$); (c) The eighth orthonormal base function ($\boldsymbol{o}_8$)

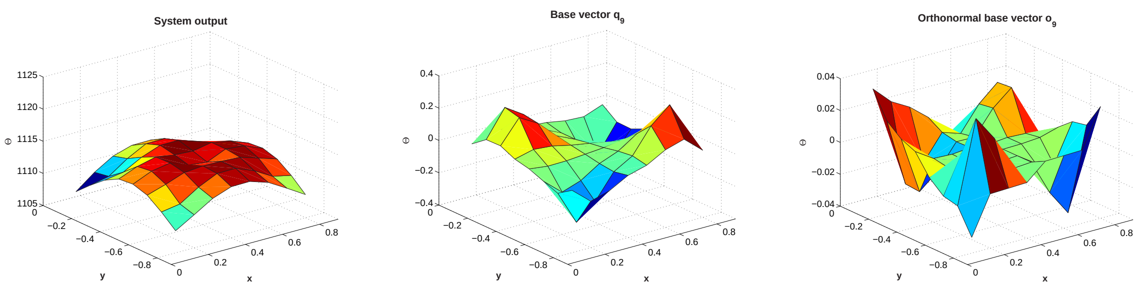

Figure 13: (a) Steady-state system output $\boldsymbol{y}$ based on the first nine states; (b) The ninth base function ($\boldsymbol{q}_9$); (c) The ninth orthonormal base function ($\boldsymbol{o}_9$)

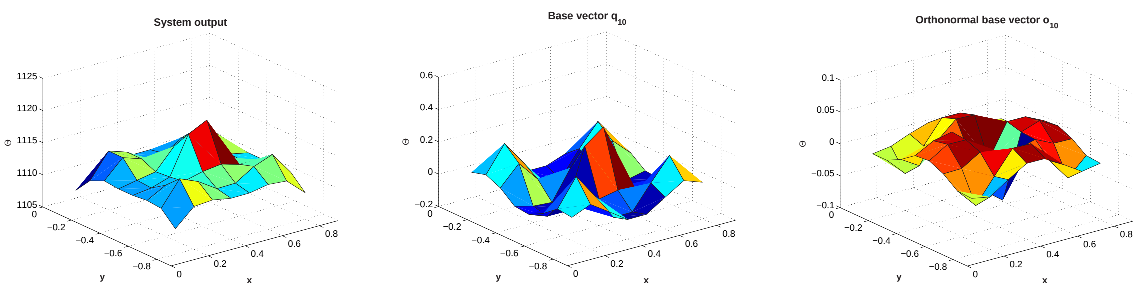

Figure 14: (a) Steady-state system output $\boldsymbol{y}$ based on the first five states; (b) The tenth base function ($\boldsymbol{q}_{10}$); (c) The tenth orthonormal base function ($\boldsymbol{o}_{10}$)

6 CONCLUSION

The state space model of the distributed parameters system described by the linear two-dimensional parabolic partial differential equation and the model reduction by the balanced truncation method are described.

The connection of the model reduction by the balanced truncation method and the solution of the partial differential equations based on the base functions is developed. The result is demonstrated on the heat transfer process example. You can observe from the figures that the base functions $\boldsymbol{q}_i$ and the orthonormal base functions $\boldsymbol{o}_i$, respectively, are similar to the Fourier decomposition of a signal (the first harmonic, the second harmonic, etc.)

Acknowledgements

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